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Note

Short Circuit Covers for Regular Matroids with a Nowhere Zero 5-Flow

URY JAMSHY

School of Mathematical Sciences, Tel Aviv University, 69978 Israel

ANDRE RASPAUD*

I. N. E. S. d'Informatique, Oued Aissi, Tizi OUzou, Algeria

AND

MICHAEL TARSI

School of Mathematical Sciences, Tel Aviv University, 69978 Israel

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The following theorem is proved: If there exists a nowhere zero 5-flow in a regular matroid on a set E, then it can be covered by circuits whose total length does not exceed $\frac{8}{5}|E|$. The constant $\frac{8}{5}$ is the smallest for which such a statement holds. © 1987 Academic Press, Inc.

Bermond *et al.* [1] have shown that the edge set E of a bridgeless graph can be covered by circuits, the total length of which is at most $\frac{5}{3}|E|$. This result is derived from the existence of a special kind of nowhere zero 8-flow (see [4]). The relation between nowhere zero flows and short circuit covers of regular matroids has been studied in [6]. The terminology we use here is that of [6], as it is defined in Sections I, II, and III of that paper. It includes the basic definitions of orientations, nowhere zero flow (NZF), etc. In particular, we use the same symbol (e.g., M) to refer to both a matroid and its element set. Following [6] we define $s(k)$ to be the smallest s such

* Present address: I.U.T. "A", Dept. Informatique, 33405 Talence Cedex, France.

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that if there exists a k-nowhere zero flow in a regular matroid M, then M can be covered by circuits, the total length of which is at most $s|M|$.

The formula obtained in Theorem I of [6] does not yield the exact value of $s(5)$. The evaluation of $s(5)$ seems to be interesting, because no counterexample is known to the following conjecture of Tutte [7]:

Every bridgeless graph has a 5-NZF

If Tutte's conjecture is true then $s(5)|E|$ is an upper bound for the length of the shortest cycle cover of a bridgeless graph. Even if Tutte's conjecture is false, the existence of a 5-NZF has been proved for several families of graphs, and a knowledge of $s(5)$ provides an upper bound for the length of a shortest cycle cover for these graphs. The interesting cases are, of course, the families for which a 4-NZF does not always exist, such as the bridgeless graphs which contain a Hamiltonian path (the existence of a 5-NZF for this family is proved in [3]). In [6] $s(5)$ is shown to lie in the closed interval $[\frac{8}{5}, \frac{5}{3}]$. Recently one of us [5] has improved that estimate, showing $s(5) \leqslant \frac{49}{30}$. In this work we completely settle that problem, proving

THEOREM 1. $s(5) = \frac{8}{5}$.

For the proof of Theorem 1 we first state several definitions and lemmas.

Let M be a regular matroid and a, b, f three functions $f: M \to \mathbb{R}$, $a, b: M \to \mathbb{Z}$ ($\mathbb{R}$, $\mathbb{Z}$ denote the additive groups of real numbers and integers, respectively), such that f is a flow in an orientation $W(M)$ of M and for every $x \in M$, $a(x) \leqslant f(x) \leqslant b(x)$. We say in this case that *f is a permissible flow under the constraints a, b.*

The following result is obtained from the work of Tutte [8], Hoffman and Kruskal [2, Theorem 2].

LEMMA 1. *Let $f: M \to \mathbb{R}$ be a permissible flow in an orientation $W(M)$ under the constraints a, b. Then there exists an integer flow $f': M \to \mathbb{Z}$ in the same orientation, permissible under the same constraints.*

Let $\lfloor y \rfloor$ denote the largest integer not greater than y, and $\lceil y \rceil$ denote the smallest integer not smaller than y. The main tool we use to prove Theorem 1 is the following simple consequence of Lemma 1:

LEMMA 2. *Let f be a Z-flow in an orientation $W(M)$ of a regular matroid M and let c be any non-zero real number. There exists a Z-flow f' in the same orientation $W(M)$, such that $f'(x) = \lfloor f(x)/c \rfloor$ or $f'(x) = \lceil f(x)/c \rceil$, for every $x \in M$.*

Proof. Define $f/c: M \to \mathbb{R}$ by $(f/c)(x) = f(x)/c$. Obviously f/c is a permissible flow under the constraints $a = \lfloor f/c \rfloor$, $b = \lceil f/c \rceil$. According to

Lemma 1 there exists a Z-flow f' permissible under the same constraints, but the only integers in the intervals defined by these constraints are $a(x)$ and $b(x)$ themselves. ∎

Let f be a 5-NZF in an orientation $W(M)$ of a regular matroid M. Define for $i \in \{1, 2, 3, 4\}$ $M_i(f) = \{x \in M \mid |f(x)| = i\}$.

LEMMA 3. *Let f be a* 5-NZF *in an orientation $W(M)$ of a regular matroid M. There exists a positive* 5-NZF $g\colon M \to \{1, 2, 3, 4\}$ *in an orientation $W'(M)$ of M, with $|M_1(g) \cup M_4(g)| \geqslant \frac{1}{2}|M|$.*

Proof. Apply Lemma 2 to f with $c = 2$. A flow f' is obtained such that $|f(x)| = 4 \Rightarrow |f'(x)| = 2$, $|f(x)| = 3 \Rightarrow |f'(x)| = 2$ or $|f'(x)| = 1$, $|f(x)| = 2 \Rightarrow |f'(x)| = 1$ and $|f(x)| = 1 \Rightarrow |f'(x)| = 1$ or $|f'(x)| = 0$. For every $x \in M$, $f'(x)$ is negative or positive only if $f(x)$ is negative or positive, respectively. It turns out that $h = 5f' - 2f$ is a 5-NZF in $W(M)$ and, $M_1(h) \cup M_4(h) = M_2(f) \cup M_3(f)$. Hence $|M_1(f) \cup M_4(f)| + |M_1(h) \cup M_4(h)| = |M|$.

Let g' be either f or h, chosen so that $|M_1(g') \cup M_4(g')| \geqslant \frac{1}{2}|M|$, and let g, $W'(M)$ be obtained from g', $W(M)$ by reversing the orientation of all $x \in M$ for which $g'(x)$ is negative. Then g, $W'(M)$ satisfy the lemma. ∎

LEMMA 4. *Let M be a regular matroid with a* 5-NZF. *Then there exists a positive* 5-NZF g, *in some orientation of M, such that $|M_1(g)| \geqslant \frac{2}{5}|M|$.*

Proof. Let f be a 5-NZF in M. By Lemma 3 we assume that f is positive and that $|M_1(f) \cup M_4(f)| \geqslant \frac{1}{2}|M|$. By applying Lemma 2 to f with $c = 2$, we obtain $f'\colon M \to \{0, 1, 2\}$, which satisfies $f'(x) = \lfloor f(x)/2 \rfloor$ or $\lceil f(x)/2 \rceil$. Another flow in M with the same property is $(f - f')$. We now apply Lemma 2 with $c = 2$ to f' to obtain $f_1 : M \to \{0, 1\}$. Define $f_2 = f' - f_1$. We obtain f_3 and $f_4 = (f - f') - f_3$ similarly by applying Lemma 2 with $c = 2$ to $(f - f')$. Define for $i \in \{1, 2, 3, 4\}$, $f'_i = 5f_i - f$. Then f'_i is a 5-NZF and $M_1(f'_i) \cup M_4(f'_i) = M_1(f) \cup M_4(f)$. From the definitions, for each $i \in \{1, 2, 3, 4\}$ $x \in M_4(f) \Rightarrow f_i(x) = 1 \Rightarrow f'_i = 1$. If $x \in M_1(f)$ then there is exactly one i for which $f_i(x) = 1$ and $f_j(x) = 0$ for $j \neq i$, which implies $f'_i(x) = 4$ and $f'_j(x) = -1$ for $j \neq i$.

Summing these equalities over $f, f'_1, f'_2, f'_3, f'_4$, we get

$$|M_1(f)| + \sum_{i=1}^{4} |M_1(f'_i)| = 4|M_1(f)| + 4|M_4(f)|.$$

Hence there exists at least one $g' \in \{f, f'_1, f'_2, f'_3, f'_4\}$, such that $|M_1(g')| \geqslant \frac{4}{5}|M_1(f) \cup M_4(f)|$. Our assumption that $|M_1(f) \cup M_4(f)| \geqslant \frac{1}{2}|M|$, implies that $M_1(g') \geqslant \frac{2}{5}|M|$. By reversing the orientation of the elements for which g' is negative we obtain the required positive flow g. ∎

Proof of Theorem 1. Let f be a 5-NZF in M. By Lemma 4 we may assume that f is positive and $|M_1(f)| \geqslant \frac{2}{5}|M|$. We construct f_1, f_2, f_3, f_4 exactly as we did in the proof of Lemma 4. For $1 \leqslant i \leqslant 4$, we denote the support of f_i by σ_i. Since f_i is a 2-flow, σ_i is a cycle of M. An element of $M_k(f)$ belongs to exactly k of the σ_i's. Thus, $C_0 = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ is a cycle cover for M and it covers $M_k(f)$ exactly k times. Take $x \in M$ such that $f(x) = 2$. From the way we constructed the f_i's (see the proof of Lemma 4) it follows that $\sigma_3 \cap \sigma_4 \subset \sigma_1 \cup \sigma_2$, for if $x \in \sigma_3 \cap \sigma_4$ then $f(x) \geqslant 3$ and so $x \in \sigma_1 \cup \sigma_2$. Thus if σ_3 and σ_4 are replaced in C_0 by their symmetric difference $\sigma_3 \nabla \sigma_4$ we still have a cycle cover of M: $C_1 = \{\sigma_1, \sigma_2, \sigma_3 \nabla \sigma_4\}$. C_1 covers $M_1(f)$ once and $M_2(f) \cup M_4(f)$ twice. $x \in M_3(f)$ is covered either once (if $x \in \sigma_3 \cap \sigma_4$) or three times (if $x \in \sigma_1 \cap \sigma_2$). By a similar analysis of the cycle cover $C_2 = \{\sigma_1 \nabla \sigma_2, \sigma_3, \sigma_4\}$, we find that every $x \in M - M_1(f)$ belongs to exactly 4 of the 6 cycles of $C_1 \cup C_2$, while $x \in M_1(f)$ belongs to exactly 2 of these cycles. It follows that $l(C_1) + l(C_2) = 4|M| - 2|M_1(f)|$. For either $C = C_1$ or $C = C_2$, $l(C) \leqslant 2|M| - |M_1(f)|$ and by our assumption that $|M_1(f)| \geqslant \frac{2}{5}|M|$ we get $l(C) \leqslant \frac{8}{5}|M|$, which implies $s(5) \leqslant \frac{8}{5}$. On the other hand $\frac{8}{5}$ is proved in [6] to be a lower bound for $s(5)$, and hence $s(5) = \frac{8}{5}$. ∎

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